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CCT 602: Creative Thinking
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Final Project: Lessons in High School Geometry

Lesson 1: the Nouns of Geometry

Learning Objectives:
After this lesson, the student should be able to:

Give and explain a definition of point, line, and straight line.

Materials:
A blackboard and chalk (or equivalent)
The Elements, Book I (Euclid)

Background Assignment:
Before beginning this lesson, students should have read *The Elements*, Book I: Definitions 1–4.

1. Lesson Model

This is the first of three lessons for a high school geometry course. The constant emphasis of these lessons is on the method of instruction, mathematical flexibility, and mathematical rigor. The approach is unified; that is, the lessons draw on material and techniques that are both common and uncommon to standard high school geometry texts. The lesson guide does not shy away from material from other mathematical subfields if it is appropriate to include it.

The structure of these plans lists a series of guiding questions. Because responses will vary from classroom to classroom, no definitive answers have been provided. Instead, the onus is on the teacher to guide the classroom according to the interests and abilities of the students. The questions are designed so that one path is highlighted over others. It is up to the instructor to decide which line of reasoning the classroom discussion to follow. Short explanations are given for items which are not normally covered in high school curricula. However, remember that these notes are merely suggestions and not holy writ. The instructor is encouraged to incorporate resources that match the individual character of his/her classroom.

2. Lesson Guide

Goal: To motivate the definition of a point.
Question: What do mathematicians mean when they talk about a point?
Comments:
Students will often give answers that refer to a "position" or coordinates or make use of an ambient or embedded space. While useful in some contexts, the beauty of Euclid's program is that it is *coordinate independent*. Stress that coordinates are merely a system of labels that humans use to make the math more tangible and that measurements are meaningless on their own.

Goal: To demonstrate that coordinate systems are a convenience rather than a necessity.
Question: Which is the correct way to measure: using inches or centimeters?

Euclid I, D1: A **point** is that which has no part.

Goal: To investigate the language of Euclid.
Question: What did Euclid say is the definition of a point?
Comments:

Euclid's attempt at a definition is really no better. In fact, it may even be less satisfying than ours.

Euclid I, D2–3: A **line** is a breadthless length, the extremities of which are two points.

Euclid I, D4: A **straight line** is a line which lies evenly with the points on itself.

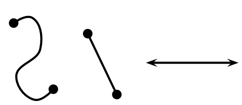
Goal: To investigate the language of definitions.
Question: How should we define a line? How does Euclid define a line?
Comments:

Students may want to define a line to be the shortest distance between two points. At this point we do not have a concept of measurement. It is impossible to distinguish the shortest path between two points without having a computational definition of length. If at this point, the teacher wishes to introduce concepts from metric geometry, he is invited to do so. These notes will remain within absolute geometry, which, as was hinted at before, is coordinate independent.

Also note that Euclid's definition does not exclude the possibility for a line to be curved. And because the extremities of a line are points, Euclid's lines are modern day line *segments*.

3. Commentary

Definitions. The questions above have been designed to get students thinking about how to come up with definitions. Often mathematical formulæ and facts are presented to students justification. Mathematics is one part exploration, one part invention. Because there is no canonical language with which to describe mathematical objects, mathematicians are put to the task of defining mathematical structures before they can embark on the discovery process of theorem proving. Definition formation is akin to problem formulation. A transformation of (equivalent) definitions can render an intractable task obvious. It has been said that Gödel claimed that ninety percent of the work is done by the definition. These lessons force the students to think creatively to come up with appropriate definitions. The teacher should prompt critical thinking with introspective follow up questions which extract the consequences of the students' thoughts into plain sight.



Three line candidates

During this lesson the teacher could lead a discussion on the nature of good definitions. Some things to keep in mind are: (1) whether your definition captures the properties you mean it to; (2) whether your definition is broad enough to include interesting examples; (3) whether your definition is narrow enough so that the set of focus is manageable. For example, ask the students why modern mathematicians chose to work with only straight lines first and then work up to curvy lines. Are they right to restrict the definition of line? To the left are some potential lines. It may be useful to ask students which and why, if any, of these figures qualifies as a line.

Coordinate Systems. Coordinate systems are a matter of great confusion. Students will want to use numbers to quantify position straight away. For the uninitiated, mathematics is a collection of formulæ, and without numbers it is hard for high school students to apply formulæ.

Intrinsic versus extrinsic geometry

Coordinate-independent approaches shift attention from measurement to the underlying geometric objects. Students are often shocked to learn that the same point can be labeled with two different coordinates. For example, decide on a fixed starting point called the origin. The directions “Start at the origin. Go one mile to the right and up one mile” and “Start at the origin. Turn 45 degrees counterclockwise and walk $\sqrt{2}$ miles” bring you to the same place. We can rightly identify $(1, 1)$ in the first coordinate system and $(45°, \sqrt{2})$ in the second system. Even though the labels look the different, they describe the same point. The point is completely unchanged by a change in label. Coordinate systems are not a part of the geometry. At this stage, it is important to allow only those definitions which are coordinate independent. Definitions which do not rely on a coordinate system are called *extrinsic*; the ones we espouse here are *intrinsic.*

4. Homework

Students should write up definitions for concepts discussed in class. A full definition will include a verbal description with an accompanying picture. Additionally, each definition should have a graphic non-example. Non-examples will satisfy parts of the definition but not all.

To get a sense of which geometric knowledge students have brought to the classroom, an instructor may ask students to illustrate (without words or conventional mathematical symbols) other geometric concepts which were not mentioned in this lesson plan. Examples include: surface area, polygon, hypotenuse, similar, congruent, circumference, bisector, and volume. Of course, use this list merely as a springboard.

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Lesson 2: the Verbs of Geometry

Learning Objectives:
After this lesson, the student should be able to:

- Give and explain a definition of circle.
- Given and explain definition of an angle (right, obtuse, acute).
- List the first four of Euclid's postulates.
- Explain what they mean and why they could be important.
- Reason in which ways the postulates do or do not capture the geometry of the blackboard.

Materials:
A blackboard and chalk (or equivalent)
The Elements, Book i (Euclid)

Background Assignment:
Before beginning this lesson, students should have read *The Elements*, Book i: Definitions 8–12, 15–16; Common Notions 1–5; Postulates 1–4.

1. Lesson Model

In this, the second of three lesson for a high school geometry course, we will continue our model of Socratic questioning to investigate the first four postulates of Euclidean geometry. In the first lesson we introduced the definitions of geometry. The definitions make up the nouns of our mathematical grammar. The postulates correspond to the verbs. They describe the way in which our geometrical object act. As was the case with definitions, there is no "natural," fixed set of postulates to include. In this lesson, we will try to explain what some of Euclid's postulates mean and why he would have included them. We will not, however, discuss the fifth, so-called Parallel Postulate until later. The lesson concludes with a construction of an equilateral triangle.

2. Lesson Guide


Postulate 1

Goal: To state the first postulate of Euclidean geometry.
Postulate: *Given any two points, there exists a unique straight line that joins them.*
Question: What does Postulate 1 mean? Can you draw it?

Goal: To explore the first postulate.
Question: Does Euclid need to specify that there is a *unique* line between any two points—could there be more; are there examples when there isn't a line at all? Does the line need to be straight?
Comments:

Postulate one tells us that there are no holes in our geometry. Remember, as of right now we do not have any way to know what our geometrical space looks like. The postulates specify on what sort of plane the lines and points we defined in the last lesson exist. Imagine a flat piece of paper. Poke a hole in the center of it. It is impossible to draw a straight line joining two points that lie on a circle whose center was the point poked out.

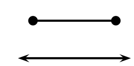
Ask the students what straight lines on a sphere could be. Does Postulate 1 hold on a sphere? What about on a cylinder?

Goal: To apply the first postulate to make a shape.
Question: What happens if I apply postulate one three times to three points? Does this method always result in a triangle; why or why not?
Comments:

This is the first of many examples to come of a *geometric construction*. The power of the postulates lies in their generative power. We can use the definitions from the first lesson as building blocks of plane geometry. The postulates are rules that allow us to put the definitions together in a consistent way.

This question might lead into a discussion about directedness. Is a line which starts at point A and joins point B the same as the line which travels from B to A? When the distinction is useful, the directed line segment is called a *ray*. If it is finite, then it is called a *vector*. Both are often denoted $\overrightarrow{AB}$, where A is the starting point and B is the endpoint. The instructor might ask if the distinction is necessary. In which cases would direction matter? In which cases is direction unimportant? Likewise, is triangle $\triangle ABC$ the same as $\triangle ACB$? The concept of a ray is revisited with the introduction the next postulate.

Goal: To state the second postulate of Euclidean geometry.
Postulate: *A line segment may be extended indefinitely in either direction.*
Question: What does the second postulate mean; can you draw it?



Postulate 2

Goal: To explore the second postulate.
Question: What does "in either direction" mean? How many directions are there? Should you always be able to extend in every possible direction? Why or why not?
Comments:

Think about races which do not end at the start line, air flights, bus routes, boat trips, or whatever else is socially relevant. Velocity is an archetypal example of a vector.

Postulate 2 tells us that our geometry is infinite. In technical math talk, postulate 2 shows that the Euclidean plane is *non-compact*. Surfaces which are compact are of finite extent. Because lines can be extended indefinitely, the Euclidean plane cannot be finite. That is why it is sometimes called the geometry of the infinite blackboard. Not all geometries are infinite. On the earth, there is a limit on the largest circle (it's the equator).

Combined with Postulate 1, the first two of Euclid's postulates give us the straightedge part of compass and straightedge geometry. As we have noted before, a straightedge is not the same thing as a ruler, though. Nothing in either of the postulates can be used to give a sense of measurement. A sense of length can be based off of a given yardstick. But lengths are not absolute, they are only relative to the unit we choose. While lengths are not absolute, ratios are. In subsequent lessons, the instructor would do well to introduce some of the classic ratio results in Euclidean geometry. It would also be reasonable to do so here.

Goal: To state a definition of a circle.
Question: How does Euclid define a circle? Does it make sense? Are there other

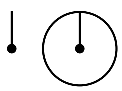
Euclid I, D15–16: A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point—called the **center**—among those lying within the figure equal one another.

ways to define it?

Comments:

Before we can introduce the compass to our geometry, we first need to define what a compass will draw. To demonstrate that we can construct a circle without knowing how to measure, the teacher can ask students each to cut a piece of string to a length that they individually find attractive. Tie a pencil to the end of the string. Hold the free end to a piece of paper and trace out the figure that results when the string is kept taut. The string was of a given length but unknown.

By the way, the Ancient Greeks planted a stake in the ground and used a taut string to draw circles. Two thousand years later, it's still the best way.



Postulate 3

Goal: To state the third postulate of Euclidean geometry.

Postulate: *Given a point and a length, there exists a circle with that point as its center and that length as its radius.*

Question: What does Postulate 3 mean? What does it look like?

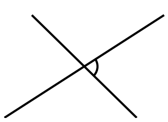
Goal: To explore the third postulate.

Question: What new information or new tools are included in Postulate 3?

Comments:

Students who have had a class that covers the conic sections can participate in a discussion related to polynomials. Circles bump up the degree of the geometry. A point is of zeroth degree. A line is of first order. A circle is of second order, a quadratic. Postulate 3 allows for geometric constructions up to degree two. This is the reason why the Greeks were unable to solve the Delian Problem: to double the cube. The solution requires the construction of a length proportional to $\sqrt[3]{2}$. Intersecting lines provide the tools to add, subtract, multiply, and divide. The circle gives a method for the extraction of a square root. The cube root, however, is impossible within the framework of Euclidean geometry.

If students have not seen algebraic techniques, it is easier to see Postulate 3 adds a compass to our straightedge and compass geometry. Later on, they will learn how to produce square lengths and square roots—probably using a circle or semi-circle.



An angle

Euclid I, D8: An **angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

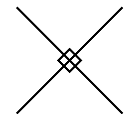
Goal: To motivate the definition of angle.

Question: What happens when we intersect two lines? How do we talk about the space in between them? How can we tell between different types of intersections? Are they all the same; how could or do they differ?

Comments:

A good discussion should single out right angles as a special case of angle. Special care must be made when talking about equality. No where does Euclid actually tell us what equal means. In the common notions, Euclid lays an axiomatic framework for equality. These axioms apply to any *equivalence relation* in general. Angular equality is a special type of equivalence relation. However, Euclid leaves it up to us to figure out what sets angular equality apart from other types of measurable quantities, like length. Angular distance is somewhat trickier for students to grapple with than linear distance. In Lesson 3 we will try to couch angular distance within a framework of motion: in fact, when investigating the *rigid motions* of an equilateral triangle.

Goal: To understand the fourth postulate of Euclidean geometry.
Postulate: *All right angles are equal to one another.*
Question: Why does Euclid include the fourth postulate; isn't it tautological—a restatement of the definition?
Comments:



Four right angles

Euclid I, D10: When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

The key here is the qualification "all." According to the definition, right angle is only a *local* concept. That is, it only takes into account information about the geometry very near to the intersection of two lines. It could be the case that the rules that govern one part of the Euclidean plane are actually different than the rules for another part. While this concept may be new to many students in math, it is a very common everyday experience. Laws vary from town to town, state to state, and country to country. In the exact same way, the geometry near the corner of a square is very different than the geometry found well within the middle of one of the sides of a square.

To see how right angles could differ from place to place, consider an American football. To make equal pairs of angles using two lines in the middle of the ball, you might use very tiny lines which meet at 90°. However, go to one the pointed ends of the football and the space between the lines might only have to be about 60°. Here is a surface that admits different kinds of right angles on different parts of its interior! Euclid does not want that sort of thing to happen.

Postulate 4 smoothes out the Euclidean plane. Because all right angles need to be the same regardless of their location in the plane, we've enforced some very rigid restrictions on the way the Euclidean plane can curve. However, we still haven't flattened out the plane. We will need the fifth, sometimes called Parallel, postulate to guarantee that the Euclidean plane is flat like a blackboard. Possible departures from this point include hyperbolic and elliptic geometry. Other fruitful (and related) plane geometries include Minkowski, cohyperbolic, doubly hyperbolic, and Galilean.[1] Results that rely only on Postulates 1–4 comprise what mathematicians call *absolute geometry*.

Absolute Geometry

In this set of lessons we will defer the introduction of the Parallel Postulate, which is of historical controversy, until much later.

3. Commentary

It is important for students to be able to understand why someone would choose a set of axioms. It is equally important for students to know that there is no correct set of axioms. Many people will ignorantly claim that logic picks out a distinguished, good set of rules. That is simply not the case. Mathematicians are free to select whichever rules are most convenient and which they believe most aptly describe the phenomena they wish to study. Inclusion, exclusion, or modification of the Parallel Postulate results in several inequivalent though rich geometries.

While it was once argued that Euclidean geometry is the geometry of nature, modern research has shown that nature freely uses different geometries for varying purposes. The globe gives us an example of spherical geometry—which is orientable cousin of the elliptic plane. Cars and trains obey the rules of Galilean geometry,

[1]See the appendices of *A Simple Non-Euclidean Geometry and its Physical Basis* by I. M. Yaglom for a really excellent introduction to all nine of the classical plane geometries.

while satellites and GPS prefer Minkowski geometry. Superluminal particles—hypothetical beasts which travel faster than the speed of light—must exist in co-hyperbolic geometry if at all. Each of these geometries is as tractable and useful as Euclidean geometry.

The postulates of Euclidean geometry encode the properties of the flat plane. It is instructive to work out just how these statements achieve what they aim to. Also, it is important to view them as actions which mathematical objects may perform. Geometry is a dynamic enterprise, and a good understanding of it will take advantage of the social and kinesthetic quality geometric behavior.

In the next lesson we will put the postulates to work to construct an equilateral triangle. To do so, students ought to reason through the construction. They could do it themselves or under the supervision of teacher. I suggest the latter. At this stage the amount of information and new framework for learning can easily overwhelm the students. The instructor should then produce a guided foundation which points toward the goal. The role of the instructor ought to be facilitative rather than instructional.

4. Homework

Ask students to come up with the postulates for the geometry of a sphere. Is it possible to connect any two points on a sphere with a unique, straight line? Can lines be extended indefinitely? To answer these questions, students will first need to come up with satisfactory definitions of point, line, circle, angle, and the like. This exercise need not be restricted to a sphere. Any surface would do. Other suggested surfaces include: a right cylinder, a right cone, an ellipsoid, or a hyperboloid.

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Lesson 3: a Beginner's Story—the Equilateral Triangle

Learning Objectives:

After this lesson, the student should be able to:

Construct an equilateral triangle of a given, finite length.

Materials:

A blackboard and chalk (or equivalent)

The Elements, Book i (Euclid)

Background Assignment:

Before beginning this lesson, students should have read *The Elements*, Book i: Proposition 1.

1. Lesson Model

Lesson 3 is a walk-through of the proof to Book i, Proposition 1: the construction of an equilateral triangle. While the instructor should still lead the class through a series of questions, the structure of the lesson guide will depart from the structure used in the previous two lessons. Instead, I have written the reasoning of the proof in the form of a short narrative. The steps of the proof are illustrated in the margins. The strategy for the proof can be summed up by the mantra "Do the only useful thing you can." Proposition 1 is an exercise in using the postulates we worked so hard to develop in Lesson 2. Lesson 3 is a "real-world" application of the that theoretic framework.

2. Lesson Guide

Proposition 1. *To construct an equilateral triangle of a finite, given length.*

The Setup. Here we start with what is known. There isn't much. All that is given is a fixed, finite length.

Step 1. The problem is no one tells us what the steps are. We need to determine them. Fortunately, there aren't many options. At this point in our geometric careers we only have four options: the four postulates.

Option 1. Postulate 1 won't add much to our picture. It will only redraw the line. We need to add another point which is equidistant from the endpoints of the line segment we already have. So we move on to our next option.

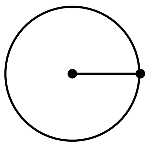
Option 2. Postulate 2 can extend the line segment. Unfortunately, it can't change the direction of the line for us. We need two new lines, one for each side of the triangle.

Option 4. Since Postulate 4 applies to angles, it doesn't apply to the case at hand. The existence of a single angle requires two lines. We only have one. It looks like a process of elimination forces us to choose Postulate 3.

Option 3. We know we need to make a circle. But how big should it be, and where should we center it? Since we're trying to make a triangle with three equal sides, it makes sense to use the line segment for the radius of the circle. That still leaves the matter of the center. There are two choices—there are, after all, only two identifiable points so far. They are the endpoints of the line segment. At this point one is no different from the other. So choose your favorite one, put down your compass, and draw!

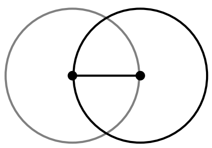
The Setup.

Step 2 Even though we've drawn a circle, the situation hasn't change drastically. There are no new lines nor new points. We need to find a third point for our triangle. The nice thing about a circle, though, is that any line we draw from the center of the circle to the circumference is the same, fixed length. In fact, it's the same length as our original line segment. That's why we drew it. The same thing will be true about a circle centered at the other endpoint.



Step 1.

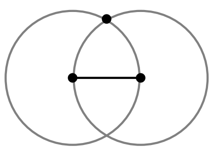
Meticulous students might want to go through the same process of elimination as above. Eventually, the amount of options at our disposal will make a thorough process of elimination unfeasible. For that reason, we need to develop an intuition. A basic strategy asks "What do I want?" and "How can I get it?"



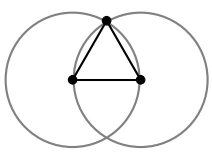
Step 2.

In this example, we want to find a point that is as far away from one endpoint as that endpoint is away from the other. The solution set is a circle. Then we want to repeat the process for the other endpoint and look at the overlap.

Step 3. There is no postulate that allows us to find the intersection points of lines and circles. In some alternative geometries, this construction does not work because the circles would not overlap. Euclid never justifies intersections. He never assumes it explicitly. He just does it. To be precise, he really needs to add another postulate. In doing so, he's adding quite a lot to his geometry. For example, imagine adding in all the irrational numbers to the fractions to fill in the space. Historically, mathematicians added new elements to their number syster when they needed to. Fractions make division work. Irrationals give many polynomials solutions. Imaginary numbers were invented so that all polynomials with real coefficients had solutions. There are other types of numbers. Likewise, Euclid has added in different types of points.



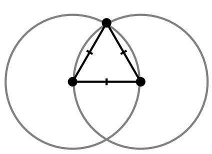
Step 3.



Step 4.

We could think about geometries whose points lie on a grid with no points in between. These planes are called lattice geometries. There are other types of point configurations. Each line Euclidean geometry is a copy of the real number line, so that circles that look like they intersect actually do intersect. We're only making one triangle, choose only one of the two points of intersection and proceed to the next step.

Step 4. Now we should give a ready application of Postulate 1. While we're at it, we might as well do it twice.



Step 5.

Step 5. For the last step, we need to explain why the triangle we constructed is an equilateral triangle—that is, why all three of the sides are the same. I leave it to you and your students as a short exercise. (*Hint:* Re-read Common Notion 1.)

3. Commentary

These lessons have begun to show the power of the axiomatic framework of Euclidean geometry. Beginning with virtually nothing—just a line segment—we were able to construct an equilateral triangle with absolute certainty that it is what we claim it to be. Few other disciplines can exercise such confidence in their results. The idea of proof is complicated and messy. It is hard to know when one has said enough. Was it completely necessary to make the remarks about intersection in Step 3? Different people will give you different answers. And they are right to.

The level of rigor required in a proof depends largely on the audience and the purpose. It might take a high school geometry class too far afield to worry about topics like *completeness* of the Euclidean line segment—though not beyond of the level of ability. Real analysis and geometry complement each other well. Step 3 of Postulate 1 gives a clear connection.

Proof strategies are as varied as the people who use them. There are more than 350 different proofs of the Pythagorean theorem alone. Instructors must be open to unfamiliar, student-offered proof strategies. It is not unprecedented that a high school student will present an entirely undiscovered proof of a well-established fact. Therefore, the instructor must foster an protected environment for the students. They must feel comfortable enough to present proofs, even if they are logically incorrect. Often incomplete proofs offer valuable insight upon which full proofs can be built. Also, it is crucial to develop an understanding that can distinguish between complete and incomplete proofs—though it is worth saying that the difference is not itself a well-defined concept.

4. Homework

The equilateral triangle is related to several other shapes. What other shapes can you construct using Postulate 1–4 and Proposition 1? Construct at least one shape different from an equilateral triangle. Examples might include a rhombus or isosceles triangle. Supply a proof of your construction. All proofs should be written in good English. Students should supply diagrams if appropriate or useful.