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Journal 3: Weak-sense, Strong-sense, and Probabilities

In linear algebra, mathematicians focus an awful lot of coordinate transformations. Good mathematics is deemed *canonical*, i.e., it comprises those results which do not rely on a choice of coordinate system or labeling. Instead, results in mathematics try to unearth those invariant truths that exist in the geometry of the underlying structure—not the ghosts which exist in labels we use to view those structures. But why did I bring that up?

Well, in math, a convenient choice of coordinate system can render a problem trivial; a bad choice can make it intractable. The difference lies in how we view the scene. One choice increases the *probability* of finding a solution; another choice decreases that probability. There is a celebrated theorem, Gleason's Theorem, which, although is a little technical in nature, I've decided to state in its full, arcane glory anyway. Basically it formalizes what I said above: if the way you look at something changes, then the probabilities of where you find yourself changes, too.[1] Then we can get into some meaty analysis with examples:

> **Theorem.** (Gleason) For a Hilbert space $\mathcal{H}$ of dimension 3 or greater, the only possible measure of the probability of the state associated with a particular linear subspace $V \subset \mathcal{H}$ will have the form $\mathrm{Tr}(\pi(V) \cdot W)$, the trace of the operator product of the projection operator $\pi(V)$ and the density matrix W for the system.

Gleason's theorem is a statement about logic, how well we can reason about things. It's a little surprising that probability and logic should be mixed up like this. After all, in Richard Paul's weak-sense, we should, as rational creatures, be able to consider a claim, mull it over a while, and eventually evaluate its correctness. To formalize things a little, we might say that we can justifiably make our way through life's logical trials with only a bivalent probability measure (either true or false). One of the immediate results of Gleason's theorem is that no such measure exists: truth has to be measured continuously. Situations can be a little true or a little false in a very real way.[2] Let's look at a few examples.

If I claim that men are taller than women and you live in New England at the time of my writing this, then you're likely to agree with me. But how certain should we be of that statement? Well, it largely depends on your interpretation. The claim was not *all* men are taller than *all* women, was it? If you interpreted it that way, then chances are you disagree with me. So, how could I ever be right, assuming that I am? The trick: probabilities.

Set up the following experiment: Stand on a well-trafficked street corner near a dimly lit alleyway during lunch time. As people pass by, kidnap one man and one woman, measure and record their heights. If the man is taller, score a tally mark under the column labeled MEN. If not, make a mark under the column labeled WOMEN. I'm a little over-confident in my claim, so we'll treat ties in favor of women. After you're done recording, let your victims go. Repeat the kidnappings until you've made ten thousand comparisons. The higher count validates (or

[1] Yes, what I mean by "find" and "where" is more than a little fuzzy. However, for now I think that's okay.

[2] Don't worry if this result doesn't pop out at you. I promise it's true with high probability.

invalidates) my claim. Note that my statement does not apply to every, individual measurement, but about all measurements *on average.* It was a statement about probabilities. For this reason, a single counter-example doesn't serve to invalidate my claim.

In general we simply don't possess a complete system of information. There's always something more we could find out more about. That's why rolling a six-sided dice is "random." The physics involved are fairly basic by today's standards. However, to measure all of the (simple) forces imparted onto the dice by your hand is so complicated that on average the results appear random. If we had perfect measurements, we could predict the outcome of each roll perfectly every time. Since we can't make perfect measurements, the results appear random.[3]

Falling short of an omniscience, we really have to construct *beliefs* about our surroundings, not certainties. It's not surprising, however, that people can act as if most things are either true or false and get away with it relatively unscathed. To get a grasp on this seeming disparity between apparent and real-with-high-probability, what we need to do is consider the probability functions associated with apparently certain events.[4]

Even though Gleason's theorem tells us that we need to be open to many, perhaps unexpected outcomes—it doesn't tell us that all outcomes occur with equal frequency. Indeed, there are a lot of things that are almost always bound to happen. The fire hydrant at the end of your street, which you saw on your way to work this morning will almost certainly be there when you come home. It almost never snows if it's not cold out. And men are taller than women—most of the time.

Moreover, it is likely that human biology prefers to use extreme contrasts to make decisions. Finer points and subtlety are lost on our reptilian brain, which is responsible for that rapid-fire fight or flight response we feel when we perceive danger. It's very possible that weak-sense reasoning is built into us. Humans simply haven't had enough time to develop a brain that relies more fully on its rational faculties.[5]

On the other hand, Paul's strong-sense reasoning nods to the fact that we often don't have all the relevant information. Many of the decisions we face today require higher-order reasoning. As such, strong-sense thinking is better suited to much of the world we live in now. We can't wait for evolution to catch up with the times. His advocating simultaneous empathy toward competing viewpoints is an awful lot like the study of coordinates transformations in linear algebra. Computationally, his method has an advantage.[6]

Strong-sense thinking has added social advantages built into it. Such an approach reduces the potency of stereotypes, which are nothing more than misapplied

[3]Even then there's order. A fair dice will fall on each of its sides about one-sixth of the time. Such happy coincidence suggests the internal determinism of the roll, or does it?

[4]Here, everything—every claim—is an event. Declare something: Class begins at 7PM, e.g. The contents of your words refer to a probabilistic event.

[5]There have been studies that show that the newer, rational brain has less of an effect on the older, reptile brain. Marketers know this fact well. See, for example, *Neuromarketing: Is There a 'Buy Button' in the Brain? Selling to the Old Brain for Instant Success* by Renvoisé and Morin.

[6]This analogy is not only intellectually compelling, it's actually applicable. See, for example, *The Geometry of Information Retrieval* by van Rijsbergen. Don't try to learn your linear algebra from van Rijsbergen, though. The text is riddled with mathematical typos.

statements of probability. I heard the following so-called positive stereotypes from a stand-up comedian on the Conan O'Brien Show many years ago:

Jews can fly. Mexicans are made out of candy.

In themselves, stereotypes aren't especially terrible. Once they've been given credence and applied inappropriately, then they start to lose their usefulness, and can even become dangerous. Even these positive stereotypes lack value, not because they are somehow hateful to their representative groups, but because statistically they lack evidence—and therefore, they lack applicability. However, don't let me fool you into thinking that no Jews can fly and that no Mexicans are made out of candy. Certainly, nothing could be further from the truth.

I believe that the trouble with weak-sense thinking is less in its atomistic foundation than it is in its commitment to polar certainties. The world is a complex thing. Gleason's theorem shows that it's perhaps even more complex than it appears on the surface.